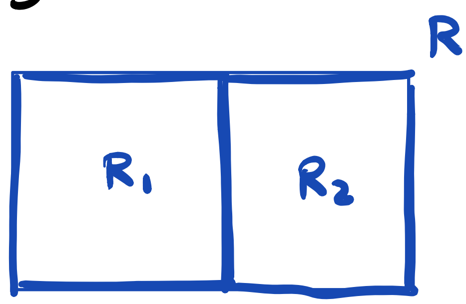
MATH 2028 - Partition of unity

GOAL : Introduce a useful "localization" tool

Recall that if $f: R \to \mathbb{R}$ is integrable on a rectangle $R \subseteq \mathbb{R}^n$ s.t.



then

$$\int_R f\,dV = \int_{R_1} f\,dV + \int_{R_2} f\,dV$$

To allow more general sub-division of a bdd domain $\Omega \subseteq \mathbb{R}^n$, it is indeed more effective to decompose the function f as

$$f = f_1 + \cdots + f_N$$

s.t. each f_i is "supported" in a smaller sub-domain of Ω. This can be achieved using a tool called "Partition of unity".

Convention: In what follows, we use $\Omega \subseteq \mathbb{R}^n$ to denote a bdd open subset whose boundary $\partial\Omega$ has measure zero.

Theorem (Partition of Unity)

For any collection $\mathcal{U} = \{U_i\}_{i \in I}$ of open subsets U_i s.t. $\Omega \subseteq \bigcup_{i \in I} U_i$, there exist a collection of C^∞ functions $\{\varphi_\alpha\}_{\alpha \in A}$ on $\mathbb{R}^n$ s.t.

(1) $\forall \alpha \in A$, $0 \leq \varphi_\alpha \leq 1$ and $\operatorname{spt} \varphi_\alpha$ is compact

(2) $\forall x \in \Omega$, $\exists$ open set V_x containing x s.t. $\operatorname{spt} \varphi_\alpha \cap V_x = \phi$ except for finitely many $\alpha \in A$. Moreover,

$$\sum_{\alpha \in A} \varphi_\alpha(x) = 1 \qquad \forall x \in \Omega$$

(3) $\forall \alpha \in A$, $\exists i \in I$ s.t. $\operatorname{spt} \varphi_\alpha \subseteq U_i$

<u>Def$^{\underline{n}}$</u>: Such $\{\varphi_\alpha\}_{\alpha \in A}$ is called a C^∞ partition of unity for $\mathcal{U}$ with compact support.

<u>Recall</u>: The support of $\varphi : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\operatorname{spt} \varphi := \overline{\{x \in \mathbb{R}^n \mid \varphi(x) \neq 0\}}$$

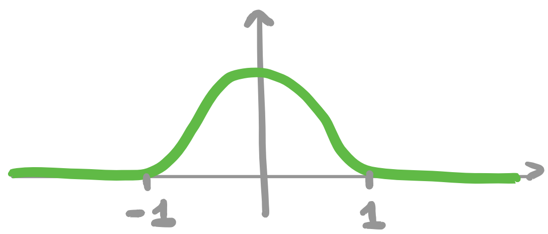
We first establish a fact that will be used in the proof of the theorem above.

FACT: Let $A \subseteq \mathbb{R}^n$ be an open set and $C \subseteq A$ be a compact subset. Then, $\exists$ C^∞ function φ defined on $\mathbb{R}^n$ s.t.

- $\varphi \geq 0$ and $\varphi(x) > 0$ $\forall x \in C$
- $\operatorname{spt} \varphi \subseteq A$

Idea of Proof:

$$f(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}}, & x \in (-1,1) \\ 0 & \text{elsewhere} \end{cases}$$



$\underline{n=1}$: $\exists$ C^∞ $f: \mathbb{R} \to \mathbb{R}$ s.t.

- $f > 0$ on $(-1, 1)$
- $f = 0$ on $\mathbb{R} \setminus (-1, 1)$

Define a C^∞ function $g: \mathbb{R}^n \to \mathbb{R}$ by

$$g(x_1, \dots, x_n) = f\left(\frac{x_1 - a_1}{\varepsilon}\right) f\left(\frac{x_2 - a_2}{\varepsilon}\right) \cdots f\left(\frac{x_n - a_n}{\varepsilon}\right)$$

THEN, $g > 0$ inside the cube of sides 2ε centered at $(a_1, \dots, a_n)$ and $g \equiv 0$ outside the cube.

Finally, cover the compact set C be finitely many cubes contained in A.

$\square$

Proof of Theorem:

- **Case I:** $\mathcal{U} = \{U_i\}_{i=1}^N$ is a finite cover of $\bar{\Omega}$

Claim 1: $\exists$ cpt $C_i \subseteq U_i$ s.t. $\bigcup_{i=1}^N \mathrm{int}(C_i) \supseteq \bar{\Omega}$

Proof: Let $B_1 = \bar{\Omega} \setminus \bigcup_{i=2}^N U_i$ which is a cpt subset of the open set U_1. Fix another cpt $C_1 \subseteq U_1$ s.t. $B_1 \subseteq \mathrm{int}(C_1)$.

Next, take $B_2 = \bar{\Omega} \setminus \left(\mathrm{int}(C_1) \cup \bigcup_{i=3}^N U_i \right)$ which is a cpt subset of U_2. Fix another cpt $C_2 \subseteq U_2$ s.t. $B_2 \subseteq \mathrm{int}(C_2)$.

Define inductively to obtain $C_1, \ldots, C_N$.

By FACT, $\exists$ C^∞ function $\psi_i : \mathbb{R}^n \to \mathbb{R}$ s.t.

- $\psi_i \geq 0$ and $\psi_i > 0$ on C_i
- $\mathrm{spt}\, \psi_i \subseteq U_i$

Note that $\sum_{i=1}^N \psi_i > 0$ on $\bigcup_{i=1}^N \mathrm{int}(C_i) \supseteq \bar{\Omega}$.

If we take a C^∞ $f : \bigcup_{i=1}^N \mathrm{int}(C_i) \to [0,1]$ with cpt support and $f \equiv 1$ on $\bar{\Omega}$. (Pf: Exercise) and $\varphi_i : \bigcup_{i=1}^{N} \operatorname{int}(C_i) \to [0,1]$ s.t.

$$\varphi_i(x) := \frac{\psi_i(x)}{\underbrace{\psi_1(x) + \psi_2(x) + \cdots + \psi_N(x)}_{>0}}$$

then $\{ f \cdot \varphi_i \}_{i=1}^{N}$ is a partition of unity

Check: • $0 \leq f, \varphi_i \leq 1 \Rightarrow 0 \leq f \cdot \varphi_i \leq 1$

• $\operatorname{spt}(f \cdot \varphi_i) = \underbrace{\operatorname{spt} f}_{\text{cpt}} \cap \operatorname{spt} \varphi_i$ is cpt

$= \operatorname{spt} f \cap \operatorname{spt} \psi_i \subseteq U_i$

• $\forall x \in \overline{\Omega}, \quad \sum_{i=1}^{N} f(x) \varphi_i(x) = \sum_{i=1}^{N} \varphi_i(x) = 1$

This proves Case I.

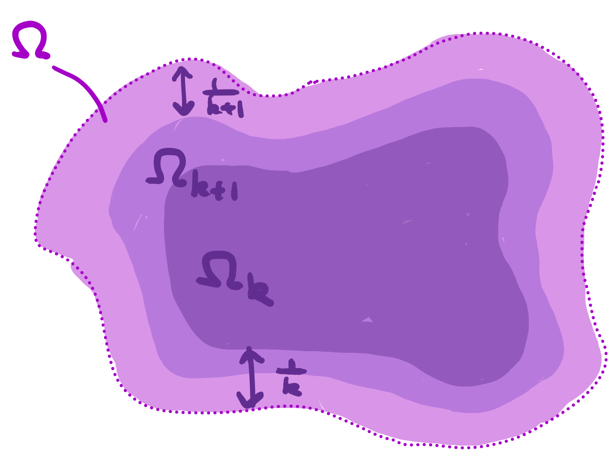
• Case II : $\mathcal{U} = \{ U_i \}_{i \in I}$ is an open cover of Ω.

Take $\Omega_k := \{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \geq k^{-1} \}$.

Note that :

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k$$

$$\underset{\text{cpt}}{\Omega_k} \subseteq \operatorname{int} \Omega_{k+1} \quad \forall k \in \mathbb{N}$$



For each $k \in \mathbb{N}$, define

$$\mathcal{U}_k := \{ U \cap (\operatorname{int} \Omega_{k+1} \setminus \Omega_{k-2}) \mid U \in \mathcal{U} \}$$

which is an open cover of the cpt set $\Omega_k \setminus \operatorname{int} \Omega_{k-1}$

Pass to a finite subcover, we can apply Case I

to obtain a partition of unity $\Phi_k = \{ \varphi_i^k \}_{i \in I_k}$

Define

$$\sigma(x) := \sum_{k=1}^{\infty} \sum_{i \in I_k} \varphi_i^k(x)$$

which is a finite sum for each $x \in \Omega$.

THEN, $\left\{ \dfrac{\varphi_i^k}{\sigma} \right\}_{i \in I_k, k \in \mathbb{N}}$ is the desired partition of unity.

$\blacksquare$

Remark: Let $K \subseteq \Omega$ be a cpt subset. Then $\exists$ at most finitely many $\alpha \in \mathcal{A}$ s.t. $\varphi_\alpha(x) \not\equiv 0$ on K.

We now see how a partition of unity can be applied to integration theory by piecing together results obtained "locally".

Theorem: Let $\Omega \subseteq \mathbb{R}^n$ be a bdd open subset with measure zero $\partial\Omega$, and $\mathcal{U} = \{U_i\}_{i \in I}$ is a collection of open sets s.t. $\Omega = \bigcup_{i \in I} U_i$. Suppose $\{\varphi_\alpha\}_{\alpha \in A}$ is a C^∞ partition of unity for $\mathcal{U}$ with compact support.

If $f : \Omega \to \mathbb{R}$ is a bdd function which is cts except on a set of measure zero, then

$$\int_\Omega f \, dV = \sum_{\alpha \in A} \int_\Omega \varphi_\alpha \cdot f \, dV$$

Remark: The sum on the R.H.S. is a countably infinite sum by the previous remark. It is indeed an infinite series which is "absolutely convergent" (c.f. MATH 2068).

Proof: Let $\varepsilon > 0$. We can choose a cpt $K \subseteq \Omega$ s.t. ∂K has measure zero and (Ex: Prove this!)

$$\mathrm{Vol}(\Omega \setminus K) < \varepsilon .$$

By the remark before the theorem, $\exists$ at most finitely many $\varphi_1, \dots, \varphi_N$ which does NOT vanish identically on K. On the other hand, by assumption $\exists\ M > 0$ s.t. $|f(x)| \leq M \quad \forall x \in \Omega$.

Then, we have

$$\left| \int_\Omega f \, dV - \sum_{i=1}^N \int_\Omega \varphi_i \cdot f \, dV \right|$$

$$\leq \int_\Omega \left| f - \sum_{i=1}^N \varphi_i \cdot f \right| dV$$

$$\leq M \int_\Omega \left(1 - \sum_{i=1}^N \varphi_i \right) dV$$

$$\leq M \int_{\Omega \setminus K} 1 \, dV = M \, \mathrm{Vol}(\Omega \setminus K) < M \varepsilon$$

Let $\varepsilon \searrow 0$ gives our desired result. $\blacksquare$